Integrable generalization of the Toda law to the square lattice

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(Received 9 July 2004; published 23 November 2004)

We generalize the Toda lattice (or Toda chain) equation to the square lattice, i.e., we construct an integrable nonlinear equation for a scalar field taking values on the square lattice and depending on a continuous (time) variable, characterized by an exponential law of interaction in both discrete directions of the square lattice. We construct the Darboux-Bäcklund transformations for such lattice, and the corresponding formulas describing their superposition. We finally use these Darboux-Bäcklund transformations to generate examples of explicit solutions of exponential and rational type. The exponential solutions describe the evolution of one and two smooth two-dimensional shock waves on the square lattice.

DOI: 10.1103/PhysRevE.70.056615 PACS number(s): 03.50.−z, 05.45.−a, 63.20.Pw, 05.50.+q

I. INTRODUCTION

The Toda lattice [1–3]

$$\frac{d^2Q_m}{dt^2} = \Delta_m e^{\Delta_m Q_{m-1}}, \tag{1}$$

where $\Delta_m f_m = f_{m+1} - f_m$ is the difference operator and $Q = Q_m(t)$ is a dynamical function on a one-dimensional lattice $(m \in \mathbb{Z})$, is one of the most famous integrable nonlinear lattice equations. It describes the dynamics of a one-dimensional physical lattice whose masses are subjected to an interaction potential of exponential type. The infinite, finite, and periodic Toda lattice (1), as well as its numerous extensions [4–12], have applications in various other physical and mathematical contexts [13–19].

In this paper we introduce the following integrable generalization of the Toda law (1) (i.e., the law characterized by an exponential interaction between nearest neighbors) to a two-dimensional (2D) lattice:

$$C_{m,n}\frac{d}{dt}\left(\frac{1}{C_{m,n}}\frac{dQ_{m,n}}{dt}\right) = \Delta_m(C_{m,n}C_{m-1,n}e^{\Delta_m Q_{m-1,n}}) + \Delta_n(C_{m,n}C_{m,n-1}e^{\Delta_n Q_{m,n-1}}), \tag{2a}$$

$$\frac{C_{m+1,n+1}}{C_{m,n}} = e^{-\Delta_m \Delta_n Q_{m,n}}, \tag{2b}$$

where $Q = Q_{m,n}(t)$ and $C = C_{m,n}(t)$ are dynamical functions on the square lattice $[(m,n) \in \mathbb{Z}^2]$ and $\Delta_m f_{m,n} = f_{m+1,n} - f_{m,n}$, $\Delta_n f_{m,n} = f_{m,n+1} - f_{m,n}$ are the difference operators in the m and n directions.

Starting with the linear five-point scheme (7) [20], in Sec. II we construct a Lax pair for Eq. (2); in Sec. III we construct the Darboux-Bäcklund transformations (DBTs) for this 2D Toda lattice, and the corresponding formulas describing the superposition of such DBTs; in Sec. IV we use these transformations to construct explicit solutions of exponential and rational type of the $(2+1)$-dimensional Toda lattice (2).

We remark that, in the literature related to integrable systems, there exist already three $(2+1)$-dimensional generalizations of the Toda lattice (1). The first one is obtained by replacing the second derivative d^2/dt^2 in Eq. (1) by the hyperbolic operator $\partial^2/\partial x\, \partial y$ [8] (this equation one can find already in the book by Darboux [21]), or by the elliptic operator $\partial^2/\partial z\, \partial \overline{z}$. In this equation, therefore, the scalar field Q depends on two continuous variables x, y and on one discrete variable m: $Q_m(x,y)$. The second generalization [5] can be viewed as a variant of the first, in which one of the two continuous variables, say x, is suitably discretized. The third generalization [6,7] is obtained by discretizing both x and y variables. In the generalization (2) we propose in this paper, instead, the scalar field Q depends on the continuous time variable t, through the Sturm-Liouville operator in the left hand side of Eq. (2), and on the two discrete indices $(m,n) \in \mathbb{Z}^2$ of the square lattice $[Q = Q_{m,n}(t)]$, through the exponential law of interaction between nearest neighbors in both m and n directions.

We remark that a $(2+1)$-dimensional generalization of the Volterra system on the square lattice has recently appeared [22].

II. THE 2D GENERALIZATION OF THE TODA LATTICE

The Lax pair (zero curvature representation) for the Toda lattice can be written in the following form [23,24]:

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$$\frac{\Gamma_m}{\Gamma_{m+1}}\phi_{m+1} + \frac{\Gamma_{m-1}}{\Gamma_m}\phi_{m-1} - F_m\phi_m = \lambda\phi_m, \tag{3a}$$

$$\frac{d\phi_m}{dt} = \frac{\Gamma_m}{\Gamma_{m+1}}\phi_{m+1} - \frac{\Gamma_{m-1}}{\Gamma_m}\phi_{m-1}, \tag{3b}$$

where λ is the constant eigenvalue of the self-adjoint three-point scheme (3a), $\Gamma_m(t)$, $F_m(t)$ are dynamical functions on the lattice, the eigenfunction $\phi_m(t,\lambda)$ solves simultaneously the Lax pair (3), and the Toda field Q is related to Γ, F in the following way:

$$F_m = -\frac{dQ_m}{dt}, \quad \Gamma_m = e^{-Q_m/2}. \tag{4}$$

Key progress toward the generalization of the Toda law (1) to a two-dimensional lattice has been recently made in [20]; in that paper, devoted to the investigation of discretizations of elliptic operators on 2D lattices admitting Darboux transformations (DTs), the following results were, in particular, established.

(1) The linear and self-adjoint five-point scheme on the star of the square lattice,

$$\mathcal{L}_5\tilde{\psi}_{m,n} := a_{m,n}\tilde{\psi}_{m+1,n} + a_{m-1,n}\tilde{\psi}_{m-1,n} + b_{m,n}\tilde{\psi}_{m,n+1} + b_{m,n-1}\tilde{\psi}_{m,n-1} - f_{m,n}\tilde{\psi}_{m,n} = 0, \tag{5}$$

a natural discretization of the self-adjoint elliptic (if $AB>0$) operator

$$(A\Psi_x)_x + (B\Psi_y)_y = \mathcal{D}\Psi \tag{6}$$

in canonical form, admits DTs.

(2) The five-point scheme (5) admits a distinguished gauge equivalent form:

$$\mathcal{L}_{SchInt}\psi_{m,n} := \frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}\psi_{m+1,n} + \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\psi_{m-1,n} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}\psi_{m,n+1} + \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\psi_{m,n-1} - F_{m,n}\psi_{m,n} = 0, \tag{7}$$

obtained from Eq. (5) via the following gauge transformation:

$$\mathcal{L}_{SchInt} = \frac{g_{m,n}}{\Gamma_{m,n}}\mathcal{L}_5\frac{g_{m,n}}{\Gamma_{m,n}}, \tag{8}$$

with $F_{m,n}=f_{m,n}(g_{m,n}^2/\Gamma_{m,n}^2)$, where g and Γ are defined by

$$a_{m,n}g_{m+1,n} = b_{m,n}g_{m+1,n},$$

$$\Gamma_{m,n} = \sqrt{a_{m,n}g_{m,n}g_{m+1,n}} = \sqrt{b_{m,n}g_{m,n}g_{m,n+1}}. \tag{9}$$

The linear problem (7), a natural 2D generalization of the linear problem (3a), satisfies the following basic properties: (i) it possesses DTs [inherited from the DTs of Eq. (5)]; (ii) it reduces, in the natural continuous limit, to the 2D Schrödinger equation

$$\psi_{xx} + \psi_{yy} + u(x,y)\psi = 0. \tag{10}$$

For these two reasons the self-adjoint five-point scheme (7) was identified in [20] as a proper "integrable" discretization of the 2D Schrödinger equation, a good starting point in the search for integrable discretizations of the nonlinear symmetries associated with the spectral problem (10) and in the search for an integrable generalization of the Toda equation to a square lattice.

The two-dimensional generalization of the Lax pair (3) proposed in this paper is indeed based on the linear problem (7), and reads

$$\frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}\psi_{m+1,n} + \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\psi_{m-1,n} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}\psi_{m,n+1} + \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\psi_{m,n-1} = F_{m,n}\psi_{m,n},$$

$$\frac{d\psi_{m,n}}{dt} = \frac{C_{m,n}}{2}\left[\frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}\psi_{m+1,n} - \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\psi_{m-1,n} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}\psi_{m,n+1} - \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\psi_{m,n-1}\right]. \tag{11}$$

It is easy to verify that this system of linear equations for the eigenfunction $\psi_{m,n}(t)$ is compatible if and only if the coefficients Γ, F, C satisfy the following nonlinear equations:

$$\frac{dF_{m,n}}{dt} = C_{m+1,n}\left(\frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}\right)^2 - C_{m-1,n}\left(\frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\right)^2 + C_{m,n+1}\left(\frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}\right)^2 - C_{m,n-1}\left(\frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\right)^2,$$

$$\frac{d\Gamma_{m,n}}{dt} = \frac{1}{2}C_{m,n}F_{m,n}\Gamma_{m,n},$$

$$\frac{C_{m+1,n+1}}{C_{m,n}} = \left(\frac{\Gamma_{m+1,n+1}\Gamma_{m,n}}{\Gamma_{m+1,n}\Gamma_{m,n+1}}\right)^2. \tag{12}$$

Equations (12) can be conveniently rewritten as the $(2+1)$-dimensional generalization (2) of the exponential interaction law of Toda, in terms of the scalar field $Q_{m,n}(t)$ defined by

$$F_{m,n}C_{m,n} = -\frac{dQ_{m,n}}{dt}, \quad \Gamma_{m,n} = e^{-Q_{m,n}/2}. \tag{13}$$

In the one-dimensional limit in which $\psi_{m,n}(t)$ depends trivially on n and the coefficients Γ, F, C do not depend on n,

$$\psi_{m,n}(t) = \phi_m(t)(-z)^n, \quad \Gamma_{m,n} = \Gamma_m, \quad F_{m,n} = F_m, \quad C_{m,n} = C_m, \tag{14}$$

it follows from Eq. (2b) that C is an arbitrary function of t independent of m, $C_m(t)=C(t)$, and Eq. (2a) reduces to

$$\frac{1}{C(t)}\frac{d}{dt}\left(\frac{1}{C(t)}\frac{dQ_m}{dt}\right) = \Delta_m e^{\Delta_m Q_{m-1}}. \tag{15}$$

Suitably rescaling t [or choosing $C(t)=1$], one finally recovers the Toda lattice (1) and its Lax pair (3), with $\lambda = z + z^{-1}$.

We first remark that the five-point schemes (11) on the star of the square lattice are the simplest and most natural generalization of the three-point schemes (3). Therefore we expect that our Toda 2D lattice (2) will be the simplest integrable generalization of Eq. (1) on the square lattice.

We also remark that the Toda 2D-lattice system (2) can be rewritten as a single equation, observing that Eq. (2b) is identically satisfied by the following parametrization:

$$\Gamma^2_{m,n} = \frac{\tau_{m,n}}{\tau_{m+1,n+1}}, \quad C_{m,n} = \frac{\tau_{m+1,n}\tau_{m,n+1}}{\tau_{m,n}\tau_{m+1,n+1}} \tag{16}$$

in terms of the single scalar field $\tau_{m,n}$. Then the system (2) reduces to the single equation

$$\begin{aligned} W[\tau_{m+1,n}\tau_{m,n+1}, W[\tau_{m+1,n+1},\tau_{m,n}]] &= \tau^2_{m+1,n}(\tau_{m,n}\tau_{m,n+2} - \tau_{m+1,n+1}\tau_{m-1,n+1}) \\ &\quad + \tau^2_{m,n+1}(\tau_{m,n}\tau_{m+2,n} - \tau_{m+1,n+1}\tau_{m+1,n-1}), \end{aligned} \tag{17}$$

where

$$W[\alpha,\beta] \coloneqq \alpha\dot{\beta} - \dot{\alpha}\beta \tag{18}$$

is the usual Wronskian of the two functions α and β. It turns out [25] that the scalar function $\tau_{m,n}(t)$ is related to the τ function of the Kadomtsev-Petviashvili hierarchy of type B (BKP hierarchy) [26]; therefore Eq. (17) gives the τ-function formulation of the $(2+1)$-dimensional Toda system (2).

If the τ function depends only on m, $\tau_{m,n}(t)=\tau_m(t)$, Eq. (17) reduces to the equation

$$\tau^2_{m+1}H[\tau_m] - \tau^2_m H[\tau_{m+1}] = 0, \tag{19}$$

where

$$H[\tau_m] \coloneqq \ddot{\tau}_m\tau_m - \dot{\tau}^2_m + \tau_{m+1}\tau_{m-1} - \tau^2_m, \tag{20}$$

implying that

$$H[\tau_m] = f(t)\tau^2_m, \tag{21}$$

with $f(t)$ an arbitrary function of t. By the change of variable $\tilde{\tau}_m(t)=\exp[-y(t)]\tau_m(t)$, with $\ddot{y}(t)=f(t)$, we recover the τ-function formulation [4] $H[\tilde{\tau}_m]=0$ of the Toda lattice (1).

We finally remark that $F,C=$const and $\Gamma=e^{(CF/2)t}$ is the trivial solution of Eq. (12); correspondingly, $Q=-FCt$ grows linearly in time. Solutions that are perturbations of this solution will exhibit such a linear blowup in time, which can be removed by introducing the change of variables $Q_{m,n}=P_{m,n}-FCt+\delta$. Then Eq. (2) becomes

$$\begin{aligned} C_{m,n}\frac{d}{dt}\left[\frac{1}{C_{m,n}}\left(\frac{dP_{m,n}}{dt}+FC\right)\right] &= \Delta_m(C_{m,n}C_{m-1,n}e^{\Delta_m P_{m-1,n}}) \\ &\quad + \Delta_n(C_{m,n}C_{m,n-1}e^{\Delta_n P_{m,n-1}}), \end{aligned}$$

$$\frac{C_{m+1,n+1}}{C_{m,n}} = e^{-\Delta_m\Delta_n P_{m,n}}. \tag{22}$$

We end this section by remarking that algebrogeometric solutions of the eigenvalue problem for a generic five-point scheme were constructed in [27].

III. DARBOUX-BÄCKLUND TRANSFORMATIONS AND THEIR SUPERPOSITION

It is a straightforward (but long) calculation to verify that the DBTs for the 2D generalization of the Toda lattice (12) and (2) read as follows:

$$\Delta_m\left(\frac{\Gamma'}{\Gamma}\theta\psi'\right)_{m,n} = \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\theta_{m,n}\theta_{m,n-1}\tilde{\Delta}_{-n}\frac{\psi_{m,n}}{\theta_{m,n}},$$

$$\Delta_n\left(\frac{\Gamma'}{\Gamma}\theta\psi'\right)_{m,n} = -\frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\theta_{m,n}\theta_{m,n-1}\tilde{\Delta}_{-m}\frac{\psi_{m,n}}{\theta_{m,n}},$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\Gamma'}{\Gamma}\theta\psi'\right)_{m,n} &= C_{m,n}\frac{\Gamma_{m-1,n}\Gamma_{m,n-1}}{\Gamma^2_{m,n}}\theta_{m-1,n}\theta_{m,n-1} \\ &\quad \times\left(\frac{\psi_{m-1,n}}{\theta_{m-1,n}} - \frac{\psi_{m,n-1}}{\theta_{m,n-1}}\right), \end{aligned} \tag{23}$$

$$\Gamma'^{\,2}_{m,n} = (\Gamma\theta)_{m-1,n-1}\frac{\Gamma_{m,n}}{\theta_{m,n}}, \tag{24a}$$

$$C'_{m,n} = \frac{(\Gamma\theta)_{m-1,n}(\Gamma\theta)_{m,n-1}}{(\Gamma\theta)_{m,n}(\Gamma\theta)_{m-1,n-1}}C_{m,n}, \tag{24b}$$

$$\begin{aligned} F'_{m,n} &= (\Gamma\theta)_{m-1,n-1}\left[\frac{1}{(\Gamma\theta)_{m-1,n}} + \frac{1}{(\Gamma\theta)_{m,n-1}}\right] + \frac{\theta_{m,n}}{\Gamma_{m,n}}\left[\left(\frac{\Gamma}{\theta}\right)_{m-1,n} \right. \\ &\quad \left. + \left(\frac{\Gamma}{\theta}\right)_{m,n-1}\right], \end{aligned} \tag{24c}$$

where $\tilde{\Delta}_{-m}f_{m,n}=f_{m,n}-f_{m-1,n}$ and $\tilde{\Delta}_{-n}=f_{m,n}-f_{m,n-1}$. In these equations, $\theta_{m,n}(t)$ is a solution of the Lax pair (11) for the coefficients $\Gamma_{m,n}$, $F_{m,n}$, $C_{m,n}$; $\psi'_{m,n}(t)$ in Eq. (23) is the transformed (via $\theta_{m,n}$) solution of the Lax pair (11), for the transformed [via Eq. (24)] coefficients $\Gamma'_{m,n}$, $F'_{m,n}$, $C'_{m,n}$.

The so-called spatial part of the above DBTs was already written in [20]; the temporal part, describing the time dependence of the transformed solution $\psi'_{m,n}(t)$ of Eq. (11), and the transformation law (24b) for the coefficient $C_{m,n}(t)$ were not.

It is well known [28] that it is possible to combine DBTs of a given integrable system, to construct superposition formulas and a permutability diagram of DBTs. For the 2D Toda lattice (12) and (2) it is possible to prove the following result (see also [25] for the Bianchi permutability diagram of the general self-adjoint scheme on the star of the square lattice). Consider a solution $(\Gamma_{m,n},F_{m,n},C_{m,n})$ of the 2D Toda lattice (12) and (2), and let $\theta^{(1)}_{m,n}$ and $\theta^{(2)}_{m,n}$ be two independent solutions of the Lax pair (11), corresponding to the coefficients $\Gamma_{m,n}$, $F_{m,n}$, $C_{m,n}$. Superimposing the two DBTs (23) with respect to $\theta^{(1)}_{m,n}$ and $\theta^{(2)}_{m,n}$, one obtains the new solution $(\Gamma^{(12)}_{m,n},F^{(12)}_{m,n},C^{(12)}_{m,n})$ of the nonlinear system (12) and (2) through the following formulas:

$$(\Gamma^{(12)}_{m+1,n+1})^2 = (\Gamma_{m,n})^2\frac{\Sigma_{m,n}}{\Sigma_{m+1,n+1}},$$

$$F^{(12)}_{m+1,n+1} = F_{m,n} + \frac{1}{\Sigma_{m,n+1}}\frac{\Gamma_{m-1,n}}{\Gamma_{m,n+1}}(\theta^{(1)}_{m-1,n}\theta^{(2)}_{m,n+1} - \theta^{(1)}_{m,n+1}\theta^{(2)}_{m-1,n}) + \frac{1}{\Sigma_{m+1,n}}\frac{\Gamma_{m,n-1}}{\Gamma_{m+1,n}}(\theta^{(1)}_{m+1,n}\theta^{(2)}_{m,n-1} - \theta^{(1)}_{m,n-1}\theta^{(2)}_{m+1,n}) = \frac{\Sigma_{m,n}\Sigma_{m+1,n+1}}{\Sigma_{m+1,n}\Sigma_{m,n+1}}\Bigg[F_{m,n} + \frac{1}{\Sigma_{m+1,n+1}}\frac{\Gamma^2_{m,n}}{\Gamma_{m+1,n}\Gamma_{m,n+1}} \times(\theta^{(1)}_{m,n+1}\theta^{(2)}_{m+1,n} - \theta^{(1)}_{m+1,n}\theta^{(2)}_{m,n+1}) + \frac{1}{\Sigma_{m,n}}\frac{\Gamma_{m-1,n}\Gamma_{m,n-1}}{\Gamma^2_{m,n}} \times(\theta^{(1)}_{m,n-1}\theta^{(2)}_{m-1,n} - \theta^{(1)}_{m-1,n}\theta^{(2)}_{m,n-1})\Bigg],$$

$$C^{(12)}_{m+1,n+1} = \left(\frac{\Gamma_{m+1,n}\Gamma_{m,n+1}}{\Gamma_{m,n}\Gamma_{m+1,n+1}}\right)^2\frac{\Sigma_{m+1,n}\Sigma_{m,n+1}}{\Sigma_{m,n}\Sigma_{m+1,n+1}}C_{m+1,n+1}, \quad (25)$$

where the function $\Sigma_{m,n}$ is obtained by integrating the first order compatible equations

$$\frac{d}{dt}\Sigma_{m,n} = C_{m,n}\frac{\Gamma_{m-1,n}\Gamma_{m,n-1}}{\Gamma^2_{m,n}}(\theta^{(1)}_{m,n-1}\theta^{(2)}_{m-1,n} - \theta^{(1)}_{m-1,n}\theta^{(2)}_{m,n-1}),$$

$$\Sigma_{m+1,n} - \Sigma_{m,n} = \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}(\theta^{(1)}_{m,n-1}\theta^{(2)}_{m,n} - \theta^{(1)}_{m,n}\theta^{(2)}_{m,n-1}),$$

$$\Sigma_{m,n+1} - \Sigma_{m,n} = \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}(\theta^{(1)}_{m,n}\theta^{(2)}_{m-1,n} - \theta^{(1)}_{m-1,n}\theta^{(2)}_{m,n}). \quad (26)$$

This scheme is often used to construct the two-soliton solution knowing the one-soliton solution of the system (in this case $\theta^{(1)}_{m,n}$ and $\theta^{(2)}_{m,n}$ are the eigenfunctions of the one-soliton solution corresponding to two different sets of parameters).

IV. SOLUTIONS OF EXPONENTIAL AND RATIONAL TYPE

The existence of DBTs is considered one of the basic properties of an integrable nonlinear system. In particular, it allows one to construct iteratively solutions from simpler solutions, via an endless procedure.

In this section we show some examples of explicit solutions of exponential and rational type of the 2D Toda lattice (2), obtained using the DBTs (24).

We consider as starting solution of the system (12) the trivial one, corresponding to F,C constants and $\Gamma = e^{(FC/2)t}$ (for this solution $Q=-FCt$) and, correspondingly, we look for an exponential solution of the Lax pair (11):

$$\psi_{m,n}(t) = e^{\alpha m+\beta n+\omega t+\delta}, \quad (27)$$

obtaining for the coefficients α,β,ω the following equations:

$$\cosh\alpha + \cosh\beta = \frac{F}{2}, \quad (28a)$$

$$\sinh\alpha + \sinh\beta = \frac{\omega}{C}. \quad (28b)$$

These equations can be interpreted in the following way: given the constants F,C, Eq. (28a) establishes a constraint between the "wave numbers" α and β; once this constraint is satisfied, Eq. (28b) gives the "dispersion relation" ω in terms of α and β.

Looking for real and nonsingular solutions of exponential type, in the following we restrict our analysis to the case $F,C,\alpha,\beta,\delta\in\mathbb{R}$, postponing the study of other possible choices to a subsequent work. Then Eq. (28) implies that $F \geqslant 4$ and that the parameters α and $\beta=\cosh^{-1}(F/2-\cosh\alpha)$ must range in the interval

$$1 < \cosh\alpha,\ \cosh\beta \leqslant \frac{F}{2} - 1, \quad F \geqslant 4 \quad (29)$$

(if $F=4$, then $\alpha=\beta=0$).

We consider now the following solution of Eq. (11):

$$\theta_{m,n}(t) = \cosh\Theta^+_{m,n}(t) + \rho\cosh\Theta^-_{m,n}(t),$$

$$\Theta^\pm_{m,n}(t) \coloneqq \alpha m \pm \beta n + \omega^\pm t + \delta^\pm, \quad \omega^\pm \coloneqq C(\sinh\alpha \pm \sinh\beta), \quad (30)$$

consisting of a suitable combination of four exponentials of the type (27), where α,β satisfy the constraint (28a), and $\rho \geqslant 0$, $\delta^\pm\in\mathbb{R}$.

Applying the DBTs (24) to this basic solution, one obtains the following dressed solution of the 2D Toda lattice (12) and (2):

$$\Gamma'^2_{m,n} = e^{-Q'_{m,n}(t)} = \frac{\cosh\Theta^+_{m-1,n-1}(t) + \rho\cosh\Theta^-_{m-1,n-1}(t)}{\cosh\Theta^+_{m,n}(t) + \rho\cosh\Theta^-_{m,n}(t)}e^{FCt},$$

$$C'_{m,n}(t) = \frac{[\cosh\Theta^+_{m-1,n}(t) + \rho\cosh\Theta^-_{m-1,n}(t)][\cosh\Theta^+_{m,n-1}(t) + \rho\cosh\Theta^-_{m,n-1}(t)]}{[\cosh\Theta^+_{m,n}(t) + \rho\cosh\Theta^-_{m,n}(t)][\cosh\Theta^+_{m-1,n-1}(t) + \rho\cosh\Theta^-_{m-1,n-1}(t)]}C. \quad (31)$$

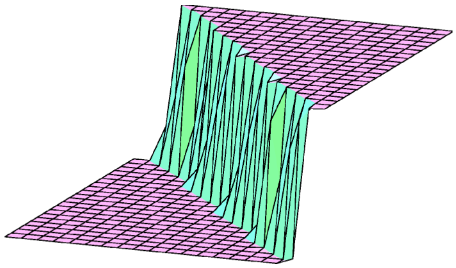

FIG. 1. (Color online) If $\rho=0$ and $\alpha,\beta\in\mathbb{R}$ $(F>4)$, the solution $Q_{m,n}+FCt$ in Eq. (31) describes a smooth 2D shock wave propagating with velocity $\mathbf{v}^{+}=-(\omega^{+}/2)\sin(2\theta)(1/\beta,1/\alpha)$. The shock front is a straight line forming the angle $-\theta$, $\theta=\tan^{-1}(\alpha/\beta)$ with the m axis. In this figure $\alpha=5$, $\beta=4$ $(F=101)$, $\delta^{\pm}=1$, $C=1$, $\rho=0$.

This solution exhibits, depending on the value of $\rho\geqslant 0$, the following different features.

(i) If $\rho=0$, we have the simplified expression

$$\Gamma_{m,n}'^{2}=e^{-Q'_{m,n}(t)}=\frac{\cosh\Theta^{+}_{m-1,n-1}(t)}{\cosh\Theta^{+}_{m,n}(t)}e^{FCt},$$

$$C'_{m,n}(t)=\frac{[\cosh\Theta^{+}_{m-1,n}(t)][\cosh\Theta^{+}_{m,n-1}(t)]}{[\cosh\Theta^{+}_{m,n}(t)][\cosh\Theta^{+}_{m-1,n-1}(t)]}C. \qquad (32)$$

This solution describes a smooth 2D shock wave (a kink); the shock front is the phase (straight) line $\Theta^{+}=$const, forming with the m axis the angle $-\theta$, with $\theta=\tan^{-1}(\alpha/\beta)$. This shock wave propagates with speed $\mathbf{v}^{+}=-(\omega^{+}/2)\sin(2\theta)\times(1/\beta,1/\alpha)$. The values of $Q'_{m,n}+FCt$ ahead and behind the shock front are, respectively, $-(\alpha+\beta)$ and $\alpha+\beta$; then the shock strength is $2(\alpha+\beta)$ (see Fig. 1).

(ii) If ρ is a finite positive number, the solution describes two smooth 2D shock waves with the following features. The phase (straight) lines $\Theta^{\pm}=$const form with the m axis the angles $\mp\theta$; they travel with speeds $\mathbf{v}^{\pm}=-(\omega^{\pm}/2)\sin(2\theta)\times(1/\beta,\pm 1/\alpha)$ and, consequently, their intersection point P travels with constant speed $\mathbf{v}_{P}=-C\big((\sinh\alpha)/\alpha,(\sinh\beta)/\beta\big)$. In this situation the two shock fronts do not coincide, as before, with the phase lines; they are now parallel to the m and n axes and intersect in P, dividing the (m,n) plane into the usual four quadrants. The values of $Q_{m,n}+FCt$ in the first, second, third, and fourth quadrants are, respectively, $\alpha+\beta$, $-\alpha+\beta$, $-(\alpha+\beta)$, and $\alpha-\beta$ (see Fig. 2).

(iii) If ρ is a very small positive parameter, $0<\rho\ll 1$, the previous two regimes combine in the following way. In the finite (m,n) plane [or, more precisely, in an inner region of the order $O(\ln(1/\rho))$], the term $\rho\cosh\Theta^{-}$ is negligible and the expression (32) is a good approximation of the solution, which then describes the single transversal shock wave of the regime (i). In the outer region, that term is not negligible anymore and the regime (ii) becomes dominant. Both ends of the transversal shock front bifurcate into two semilines par-

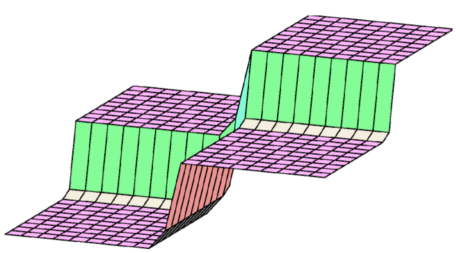

FIG. 2. (Color online) For $\alpha,\beta\in\mathbb{R}$ $(F>4)$ and $\rho=O(1)$, the solution $Q_{m,n}+FCt$ in Eq. (31) describes two shock waves with fronts parallel to the m and n axes. The intersection point P of these two fronts travels with velocity $\mathbf{v}_{P}=-C\big((\sinh\alpha)/\alpha,(\sinh\beta)/\beta\big)$. In this figure $\alpha=5$, $\beta=4$ $(F=101)$, $\delta^{\pm}=1$, $C=1$, $\rho=1$.

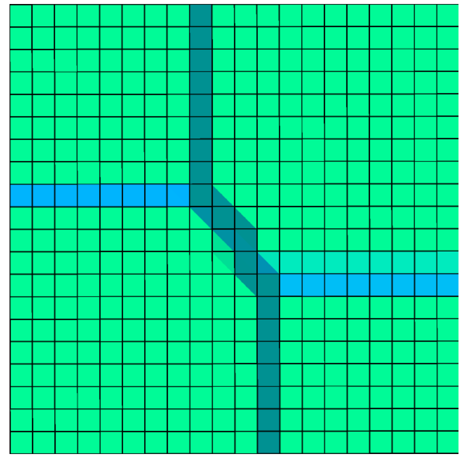

FIG. 3. (Color online) A view from the top of the solution for $\rho=10^{-7}$. In the central (finite) region, the single shock prevails; this single shock matches with the two orthogonal shocks, which prevail instead in the outer region. In this figure $\alpha=5$, $\beta=4$ $(F=101)$, $\delta^{\pm}=1$, $C=1$, $\rho=10^{-7}$.

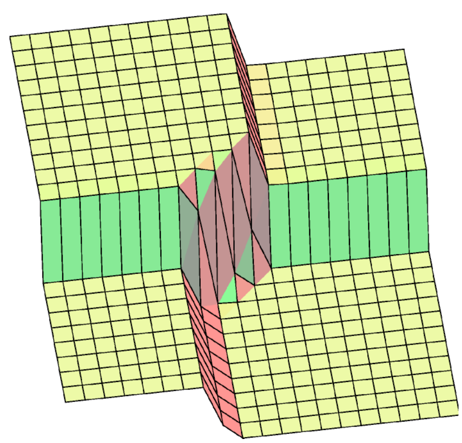

FIG. 4. (Color online) A generic view of the solution for $\rho=10^{-7}$. In the central (finite) region, the single shock prevails; this single shock matches with the two orthogonal shocks, which prevail instead in the outer region. In this figure $\alpha=5$, $\beta=4$ $(F=101)$, $\delta^{\pm}=1$, $C=1$, $\rho=10^{-7}$.

allel to the m and n axes (see Figs. 3 and 4). One could actually say that the regime (iii) is the generic one; but, for $\rho = O(1)$, the inner region is not visible, since it is smaller than a single elementary square of the square lattice (m,n). The inner region is visible if it contains at least one elementary square of the lattice. If the spacing of the square lattice is 1, a rough extimate for this condition is that $0 < \rho < \min\left(1/(|\alpha|e), 1/(|\beta|e)\right)$, where e is the Neper constant.

The possible existence of weblike structures in the inner region, typical of $(2+1)$-dimensional soliton models [29], will be explored in a subsequent work.

Starting with the trivial solution $F=4$, C const ($\Rightarrow \Gamma = e^{4Ct}$) of the system (12), it is also possible to construct rational solutions. It is straightforward to verify, for instance, that

$$\theta_{m,n}(t) \coloneqq mn + (Ct+a)m + (Ct+b)n + C^2t^2 + C(a+b)t + d \tag{33}$$

is a polynomial solution of the system (11), where a, b, d are arbitrary constant coefficients. Substituting it into the DBTs (24), one obtains a (singular) rational solution of the 2D Toda lattice. We remark that this solution could have been derived directly from the solution (31) for a suitable choice of its free parameters.

ACKNOWLEDGMENTS

This work was supported by the cultural and scientific agreements between the University of Roma “La Sapienza” and the Universities of Warsaw and Olsztyn. It was partially supported by KBN Grant No. 2 P03B 126 22.

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